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SIMPLE POPULATION MODELS WITH DIFFUSION†

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Abstract—A class of reaction-diffusion models is discussed. These models contain internal reactions (kinetics) and motion in one space dimension (diffusion). Two kinds of diffusion, random and directed, are considered. Simple models in population and epidemic theory are presented to illustrate the flexibility of the directed diffusion concept.

INTRODUCTION

We consider reaction-diffusion processes in one space dimension. These are described by N state variables $U^1, \ldots, U^N$ which are functions of time t and position x. It is helpful to think of the U^i as representing concentrations of particles of different "species" which are moving in one dimension.

We assume there are two processes taking place. One consists of internal reactions among the species which are independent of the motion. We term this the *kinetics*. The second process is that of particle motion or *diffusion*.

The basic relation governing the above situation is a set of *balance* laws. These have the form

$$U_t^i(t,x) = S^i(t,x) - Q_x^i(t,x) \quad i = 1, \ldots, N. \tag{B}$$

In these S^i represents the kinetics and Q^i is the flux so that $Q_x^i(t,x)$ represents change due to diffusion. In the situations we study (temporal and spatial homogeneity) we will have

$$S_i(t,x) = \mathcal{S}^i(U^1(t,x), \ldots U^N(t,x)). \tag{1}$$

We give some examples of kinetics later but first we describe two different diffusion mechanisms.

1. RANDOM DIFFUSION

The prototype here is chemical diffusion theory. The underlying assumption is that particles are equally likely to move right or left. This process is well known and can be studied probabilistically. The outcome is that a reasonable assumption about the flux (with temporal and spatial homogeneity) is

$$Q^i(t,x) = -K^i U_x^i(t,x). \tag{R}$$

2. DIRECTED DIFFUSION

This is analogous to fluid flow. One assumes that the motion is not random but that there is a well-determined *velocity* $v^i(t,x)$ for a particle of species i at position x and time t. Here the assumption about Q^i is

$$Q^i(t,x) = U^i(t,x)v^i(t,x). \tag{D}$$

The idea of directed diffusion in population dynamics was introduced probabilistically in [1] and deterministically in [2]. Models involving this idea were analyzed in [3] and [4]. We will not

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reproduce this analysis here but will content ourselves mainly with describing some models. Before doing so, however, we do want to indicate one quite general qualitative difference between the twp types of diffusion.

If one substitutes (R) into (B) one obtains a parabolic system, diagonalized in the diffusion, with possibly nonlinear source terms. This kind of situation has been widely studied in chemical theory.

For (D) the situation is very different. Let us substitute (1) and (D) into (B) and rewrite the result as,

$$U_t^i + v^i U_x^i = \mathscr{S}^i(U^1, \dots U^N) - v_x^i U^i. \tag{2}$$

Now introduce the characteristic curves $X^i(t\colon \bar{t}, \bar{x})$ defined by

$$\frac{\partial X^i}{\partial t} = v^i(t, X^i), X^i(\bar{t}; \bar{t}, \bar{x}) = \bar{x}. \tag{C}$$

Suppose U^i solves (2). For fixed $\bar{t}, \bar{x}$ put

$$\tilde{U}(t) = U^i(t, X^I(t\colon \bar{t}, \bar{x}))$$

$$\tilde{\mathscr{S}}^i(t, \tilde{\mathrm{U}}, \dots, \tilde{\mathrm{U}}^N) = \mathscr{S}^i(\tilde{\mathrm{U}}^1, \dots, \tilde{\mathrm{U}}^N) - v_x^i(t, X^i(t\colon \bar{t}, \bar{x}))\tilde{\mathrm{U}}^i.$$

Then (2) becomes

$$\frac{\mathrm{d}\tilde{\mathrm{U}}^i}{\mathrm{d}t} = \mathscr{S}^i(t, \mathrm{U}^1, \dots, \tilde{\mathrm{U}}^N). \tag{S}$$

Equations (3) look just like those for stationary processes in which there are no motions but the kinetics are no longer temporally homogeneous. This formal calculation gives some indication of the interaction of kinetics and diffusions for case (D).

We proceed now to describe some simple models in population dynamics including diffusion.

(1) *Single species age-dependent models*

A general setting for age dependent single species models is given in [3]. There is one state variable $U(t, x)$ and an age-density $u(t, x, a)$ with

$$U(t, x) = \int_0^\infty u(t, x, a)\,\mathrm{d}a.$$

There is a term representing deaths which is taken to have the form $-\mu(a, U(t, x))\mu$ and again there is a diffusion flux $q(t, x, a)$. The balance law here represents change with time of a "cohort" that is individuals of age aa and reads,

$$u_t + u_a = -\mu u - q_x. \tag{4}$$

To this is appended a birth law of the form,†

$$u(t, x, 0) = \int_0^\infty \beta(a, U(t, x))u(t, x, a)\,\mathrm{d}a. \tag{5}$$

This model allows the death and birth moduli to depend on "total population" U as well as age. Random diffusion here corresponds to the assumption

$$q(t, x, a) = -K_0 u_x(t, x, a). \tag{6$_\mathrm{R}$}$$

†We are assuming again temporal and spatial homogeneity in both kinetics and diffusion.

One can use directed diffusion to construct a model for species which diffuse to avoid crowding. For this we assume

$$q(t, x, a) = -K_0 u(t, x, a) U_x(t, x). \tag{6_D}$$

In [3] and [4] we obtained simpler models by simplifying the kinetics. We had three main examples:

(i) *Age independent kinetics.* Here we take $\mu = \mu_0(U)$, $\beta = \beta_0(U)$. Then (5) yields $U(t, x, 0) = \beta_0(U)U$. If we integrate (4) with respect to a from 0 to ∞ we obtain, then,

$$U_t = (\beta_0(U) - \mu_0(U))U + \begin{matrix} K_0 U_{xx} \\ K_0(UU_x)_x \end{matrix} \text{ for } \begin{matrix} (6_R) \\ (6_D) \end{matrix}. \qquad \begin{matrix} (7_R) \\ (7_D) \end{matrix}$$

Model (7_R) was introduced in [5] and is a heat equation with non-linear source. Model (7_D) was introduced in [2]. For $\beta_0 = \mu_0$ it is the equation of porous media flow.

(ii) *Exponential birth modulus.* Here one again puts $\mu = \mu_0(U)$ but takes β in the form $\beta = \beta_0(U)a\, e^{-\alpha a}$. One can again reduce this to partial differential equations of the form (B) but with three functions. Put $U^1 = U$ and

$$U^2(t, x) = \int_0^\infty d^{-\alpha a} u(t, x, a)\, da; \; U^3(t, x) = \int_0^\infty a\, e^{-\alpha a} U(t, x, a)\, da$$

Then by multiplying (4) by 1, $e^{-\alpha a}$ and $a\, e^{-\alpha a}$ and integrating one finds a set of equations of the form (B) with,

$$\mathcal{S}^1(U^1, U^2, U^3) = \beta_0(U^1)I^3 - \mu_0(U^1)U^1$$

$$\mathcal{S}(U^1, U^2, U^3) = \beta_0(U^1)U^3 - (\mu_0(U^1) + \alpha)U^2$$

$$\mathcal{S}(U^1, U^2, U^3) = U^2 - (\mu_0(U^1) + \alpha)U^3.$$

With the random diffusion (B) becomes a parabolic system. With the directed diffusion (6_D) however one has the equations

$$U_t^i = \mathcal{S}^i(U^1, U^2, U^3) + K_0(U_x^1 U^i). \tag{8}$$

Observe that the characteristic curves for all three U^i's are the same,

$$\frac{\partial X^i}{\partial t} = -K_0 U_x^1(t, X^i).$$

(iii) *Separable death modulus.* This case was studied in [4]. We assume that μ has the form $\mu = \mu_n(a) + \mu_e(U)$. We think of $\mu_n(a)$ as representing deaths due to natural causes and μ_e as representing environmental effects. We also assume β independent of U, $\beta(a, U) = \beta(a) \in L_1(0, \infty)$. What was done in [4] was to look for product solutions $u(t, a, x) = \xi(a)\mathfrak{U}(t, x)$. The result was that if $\pi(a) = \exp\{\int_0^a \mu_n(\alpha)\, d\alpha$ and r is the unique root of $\int_0^\infty \beta(a)\pi(a)\exp(-ra)\, da = 1$ then one obtains product solutions for diffusions (6) by putting $\mathfrak{Z}(a) = c\pi(a)\exp(-ra)$, c a constant and taking $\mathfrak{U}$ as a solution of

$$\mathfrak{U}_t = S(\mathfrak{U}) + \begin{matrix} K_0\mathfrak{U}_{xx} \\ K_0(\mathfrak{U}_x\mathfrak{U})_x \end{matrix}, \; S(\mathfrak{U}) = (r - \mathfrak{U}_e(\mathfrak{U}).$$

This is the same as (*i*) with β_0 replaced by *r*.

We mention now two additional models which exploit directed diffusion and present interesting mathematical problems but for which we as yet have no analysis.

(2) *A predator-prey model*

Here there are two species a prey U^1 and a predator U^2. There is a well known kinetics for this situation in the absence of diffusion. It is due to Lotka and assumes

$$S^1(U^1, U^2) = \alpha U^1 - \beta U^1 U^2, S^2(U^1, U^2) = -\gamma U^2 + \delta U^1 U^2.$$

If one assumes random diffusion (B) then again gives a parabolic system with no predator-prey interaction in the diffusion. A more realistic model would be diffusion in which the prey moves away from concentrations of predators, $v^1 = -K_1 U_x^2$, while the predator moves toward concentration of prey, $v^2 = +K_2 U_x^1$, $K_1, K_2 > 0$. Thus (B) yields,

$$U_t^1 = \alpha U^1 - \beta U^1 U^2 + K_1(U_x^2 U^1)_x;\ U_t^2 = -\gamma U^2 + \delta U^1 U^2 - K_2(U_x^1 U^2)$$

(3) *An epidemic model*

Here there are three species, susceptibles U^1, inflectives U^2 and removed (recovered) U^3. A simple kinetics for this situation in the stationary case is due to Kermack and McKendrick and is,

$$S^1(U^1, U^2, U^3) = -aU^1 U^2;\ S^2(U^1, U^2, U^3) = aU^1 U^2 - \lambda U^2$$
$$S^3(U^1, U^2, U^3) = \lambda U^2,\ a, \lambda > 0.$$

These kinetics with random diffusion were studied in [6]. With directed diffusion one can again obtain quite interesting models. As a simple example suppose infectives and removed do not move $v^2 = v^3 = 0$, while susceptibles move away from concentrations of infectives. This yields the system,

$$U_t^1 = -aU^1 U^2 + K_0(U_x^2 U^1)$$
$$U_t^2 = aU^1 U^2 - \lambda U^2,\ U_t^3 = \lambda U^2.$$

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